

AP Calculus AB/BC Review

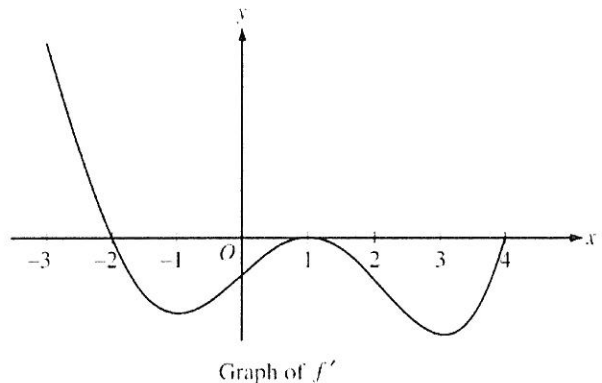
Value of a Graph

SOLUTIONS AND SCORING

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Question 5

The figure above shows the graph of f' , the derivative of a twice-differentiable function f , on the interval $[-3, 4]$. The graph of f' has horizontal tangents at $x = -1$, $x = 1$, and $x = 3$. The areas of the regions bounded by the x -axis and the graph of f' on the intervals $[-2, 1]$ and $[1, 4]$ are 9 and 12, respectively.



- (a) Find all x -coordinates at which f has a relative maximum. Give a reason for your answer.
- (b) On what open intervals contained in $-3 < x < 4$ is the graph of f both concave down and decreasing? Give a reason for your answer.
- (c) Find the x -coordinates of all points of inflection for the graph of f . Give a reason for your answer.
- (d) Given that $f(1) = 3$, write an expression for $f(x)$ that involves an integral. Find $f(4)$ and $f(-2)$.

- (a) $f'(x) = 0$ at $x = -2$, $x = 1$, and $x = 4$.
 $f'(x)$ changes from positive to negative at $x = -2$.
 Therefore, f has a relative maximum at $x = -2$.

2 : $\begin{cases} 1 : \text{identifies } x = -2 \\ 1 : \text{answer with reason} \end{cases}$

- (b) The graph of f is concave down and decreasing on the intervals $-2 < x < -1$ and $1 < x < 3$ because f' is decreasing and negative on these intervals.

2 : $\begin{cases} 1 : \text{intervals} \\ 1 : \text{reason} \end{cases}$

- (c) The graph of f has a point of inflection at $x = -1$ and $x = 3$ because f' changes from decreasing to increasing at these points.

2 : $\begin{cases} 1 : \text{identifies } x = -1, 1, \text{ and } 3 \\ 1 : \text{reason} \end{cases}$

The graph of f has a point of inflection at $x = 1$ because f' changes from increasing to decreasing at this point.

- (d) $f(x) = 3 + \int_1^x f'(t) dt$

$$f(4) = 3 + \int_1^4 f'(t) dt = 3 + (-12) = -9$$

$$\begin{aligned} f(-2) &= 3 + \int_1^{-2} f'(t) dt = 3 - \int_{-2}^1 f'(t) dt \\ &= 3 - (-9) = 12 \end{aligned}$$

3 : $\begin{cases} 1 : \text{integrand} \\ 1 : \text{expression for } f(x) \\ 1 : f(4) \text{ and } f(-2) \end{cases}$

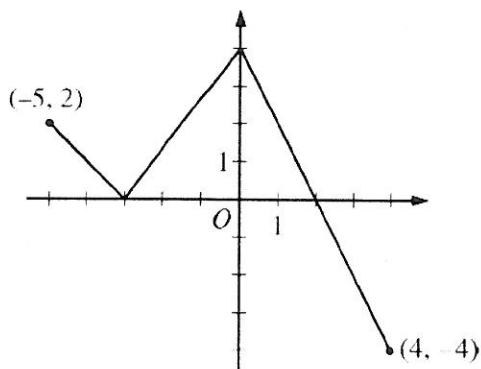
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Question 3

The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above.

Let g be the function defined by $g(x) = \int_{-3}^x f(t) dt$.

- (a) Find $g(3)$.
- (b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.
- (c) The function h is defined by $h(x) = \frac{g(x)}{5x}$. Find $h'(3)$.
- (d) The function p is defined by $p(x) = f(x^2 - x)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.



Graph of f

(a) $g(3) = \int_{-3}^3 f(t) dt = 6 + 4 - 1 = 9$

1 : answer

(b) $g'(x) = f(x)$

The graph of g is increasing and concave down on the intervals $-5 < x < -3$ and $0 < x < 2$ because $g' = f$ is positive and decreasing on these intervals.

2 : $\begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$

(c) $h'(x) = \frac{5xg'(x) - g(x)5}{(5x)^2} = \frac{5xg'(x) - 5g(x)}{25x^2}$

3 : $\begin{cases} 2 : h'(x) \\ 1 : \text{answer} \end{cases}$

$$\begin{aligned} h'(3) &= \frac{(5)(3)g'(3) - 5g(3)}{25 \cdot 3^2} \\ &= \frac{15(-2) - 5(9)}{225} = \frac{-75}{225} = -\frac{1}{3} \end{aligned}$$

(d) $p'(x) = f'(x^2 - x)(2x - 1)$

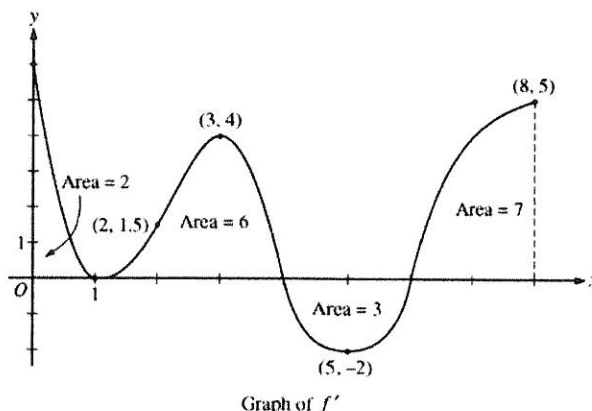
3 : $\begin{cases} 2 : p'(x) \\ 1 : \text{answer} \end{cases}$

$$p'(-1) = f'(2)(-3) = (-2)(-3) = 6$$

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Question 4

The figure above shows the graph of f' , the derivative of a twice-differentiable function f , on the closed interval $0 \leq x \leq 8$. The graph of f' has horizontal tangent lines at $x = 1$, $x = 3$, and $x = 5$. The areas of the regions between the graph of f' and the x -axis are labeled in the figure. The function f is defined for all real numbers and satisfies $f(8) = 4$.



- (a) Find all values of x on the open interval $0 < x < 8$ for which the function f has a local minimum. Justify your answer.
- (b) Determine the absolute minimum value of f on the closed interval $0 \leq x \leq 8$. Justify your answer.
- (c) On what open intervals contained in $0 < x < 8$ is the graph of f both concave down and increasing? Explain your reasoning.
- (d) The function g is defined by $g(x) = (f(x))^3$. If $f(3) = -\frac{5}{2}$, find the slope of the line tangent to the graph of g at $x = 3$.

- (a) $x = 6$ is the only critical point at which f' changes sign from negative to positive. Therefore, f has a local minimum at $x = 6$.

- (b) From part (a), the absolute minimum occurs either at $x = 6$ or at an endpoint.

$$\begin{aligned} f(0) &= f(8) + \int_8^0 f'(x) \, dx \\ &= f(8) - \int_0^8 f'(x) \, dx = 4 - 12 = -8 \end{aligned}$$

$$\begin{aligned} f(6) &= f(8) + \int_8^6 f'(x) \, dx \\ &= f(8) - \int_6^8 f'(x) \, dx = 4 - 7 = -3 \end{aligned}$$

$$f(8) = 4$$

The absolute minimum value of f on the closed interval $[0, 8]$ is -8 .

- (c) The graph of f is concave down and increasing on $0 < x < 1$ and $3 < x < 4$, because f' is decreasing and positive on these intervals.

- (d) $g'(x) = 3[f(x)]^2 \cdot f'(x)$

$$g'(3) = 3[f(3)]^2 \cdot f'(3) = 3\left(-\frac{5}{2}\right)^2 \cdot 4 = 75$$

1 : answer with justification

3 : $\begin{cases} 1 : \text{considers } x = 0 \text{ and } x = 6 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

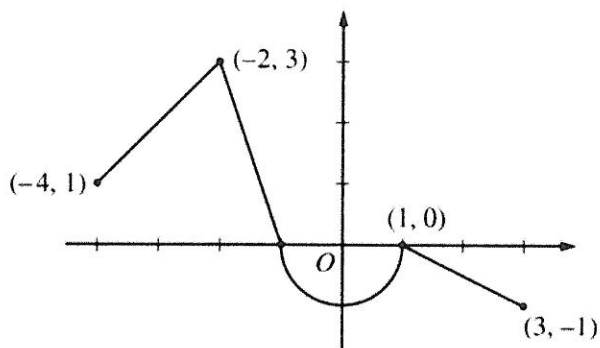
2 : $\begin{cases} 1 : \text{answer} \\ 1 : \text{explanation} \end{cases}$

3 : $\begin{cases} 2 : g'(x) \\ 1 : \text{answer} \end{cases}$

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Question 3

Let f be the continuous function defined on $[-4, 3]$ whose graph, consisting of three line segments and a semicircle centered at the origin, is given above. Let g be the function given by $g(x) = \int_1^x f(t) dt$.



Graph of f

- (a) Find the values of $g(2)$ and $g(-2)$.
- (b) For each of $g'(-3)$ and $g''(-3)$, find the value or state that it does not exist.
- (c) Find the x -coordinate of each point at which the graph of g has a horizontal tangent line. For each of these points, determine whether g has a relative minimum, relative maximum, or neither a minimum nor a maximum at the point. Justify your answers.
- (d) For $-4 < x < 3$, find all values of x for which the graph of g has a point of inflection. Explain your reasoning.

(a) $g(2) = \int_1^2 f(t) dt = -\frac{1}{2}(1)\left(\frac{1}{2}\right) = -\frac{1}{4}$

$$g(-2) = \int_1^{-2} f(t) dt = -\int_{-2}^1 f(t) dt$$

$$= -\left(\frac{3}{2} - \frac{\pi}{2}\right) = \frac{\pi}{2} - \frac{3}{2}$$

$$2 : \begin{cases} 1 : g(2) \\ 1 : g(-2) \end{cases}$$

(b) $g'(x) = f(x) \Rightarrow g'(-3) = f(-3) = 2$
 $g''(x) = f'(x) \Rightarrow g''(-3) = f'(-3) = 1$

$$2 : \begin{cases} 1 : g'(-3) \\ 1 : g''(-3) \end{cases}$$

- (c) The graph of g has a horizontal tangent line where $g'(x) = f(x) = 0$. This occurs at $x = -1$ and $x = 1$.

$g'(x)$ changes sign from positive to negative at $x = -1$.
 Therefore, g has a relative maximum at $x = -1$.

$g'(x)$ does not change sign at $x = 1$. Therefore, g has neither a relative maximum nor a relative minimum at $x = 1$.

$$3 : \begin{cases} 1 : \text{considers } g'(x) = 0 \\ 1 : x = -1 \text{ and } x = 1 \\ 1 : \text{answers with justifications} \end{cases}$$

- (d) The graph of g has a point of inflection at each of $x = -2$, $x = 0$, and $x = 1$ because $g''(x) = f'(x)$ changes sign at each of these values.

$$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{explanation} \end{cases}$$

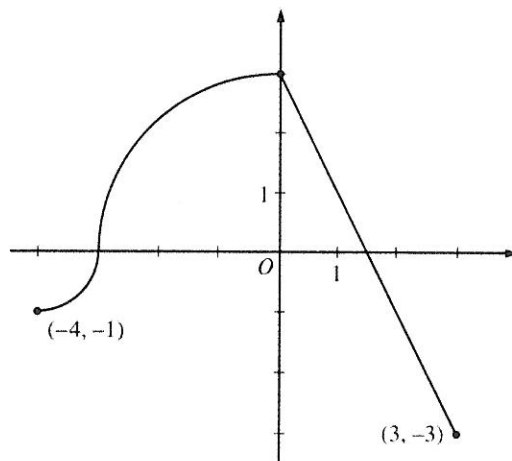
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2011 SCORING GUIDELINES

Question 4

The continuous function f is defined on the interval $-4 \leq x \leq 3$.

The graph of f consists of two quarter circles and one line segment, as shown in the figure above.

Let $g(x) = 2x + \int_0^x f(t) dt$.



- (a) Find $g(-3)$. Find $g'(x)$ and evaluate $g'(-3)$.
- (b) Determine the x -coordinate of the point at which g has an absolute maximum on the interval $-4 \leq x \leq 3$. Justify your answer.
- (c) Find all values of x on the interval $-4 < x < 3$ for which the graph of g has a point of inflection. Give a reason for your answer.
- (d) Find the average rate of change of f on the interval $-4 \leq x \leq 3$. There is no point c , $-4 < c < 3$, for which $f'(c)$ is equal to that average rate of change. Explain why this statement does not contradict the Mean Value Theorem.

(a) $g(-3) = 2(-3) + \int_0^{-3} f(t) dt = -6 - \frac{9\pi}{4}$
 $g'(x) = 2 + f(x)$
 $g'(-3) = 2 + f(-3) = 2$

3 : $\begin{cases} 1 : g(-3) \\ 1 : g'(x) \\ 1 : g'(-3) \end{cases}$

(b) $g'(x) = 0$ when $f(x) = -2$. This occurs at $x = \frac{5}{2}$.
 $g'(x) > 0$ for $-4 < x < \frac{5}{2}$ and $g'(x) < 0$ for $\frac{5}{2} < x < 3$.
 Therefore g has an absolute maximum at $x = \frac{5}{2}$.

3 : $\begin{cases} 1 : \text{considers } g'(x) = 0 \\ 1 : \text{identifies interior candidate} \\ 1 : \text{answer with justification} \end{cases}$

(c) $g''(x) = f'(x)$ changes sign only at $x = 0$. Thus the graph of g has a point of inflection at $x = 0$.

1 : answer with reason

(d) The average rate of change of f on the interval $-4 \leq x \leq 3$ is
 $\frac{f(3) - f(-4)}{3 - (-4)} = -\frac{2}{7}$.

2 : $\begin{cases} 1 : \text{average rate of change} \\ 1 : \text{explanation} \end{cases}$

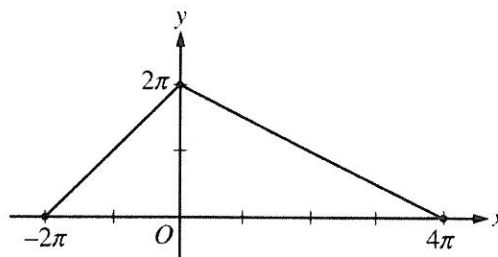
To apply the Mean Value Theorem, f must be differentiable at each point in the interval $-4 < x < 3$. However, f is not differentiable at $x = -3$ and $x = 0$.

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2011 SCORING GUIDELINES (Form B)

Question 6

Let g be the piecewise-linear function defined on $[-2\pi, 4\pi]$

whose graph is given above, and let $f(x) = g(x) - \cos\left(\frac{x}{2}\right)$.



Graph of g

- (a) Find $\int_{-2\pi}^{4\pi} f(x) dx$. Show the computations that lead to your answer.
- (b) Find all x -values in the open interval $(-2\pi, 4\pi)$ for which f has a critical point.
- (c) Let $h(x) = \int_0^{3x} g(t) dt$. Find $h'\left(-\frac{\pi}{3}\right)$.

$$\begin{aligned} \text{(a)} \quad \int_{-2\pi}^{4\pi} f(x) dx &= \int_{-2\pi}^{4\pi} \left(g(x) - \cos\left(\frac{x}{2}\right) \right) dx \\ &= 6\pi^2 - \left[2\sin\left(\frac{x}{2}\right) \right]_{x=-2\pi}^{x=4\pi} \\ &= 6\pi^2 \end{aligned}$$

2 : $\begin{cases} 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

$$\text{(b)} \quad f'(x) = g'(x) + \frac{1}{2}\sin\left(\frac{x}{2}\right) = \begin{cases} 1 + \frac{1}{2}\sin\left(\frac{x}{2}\right) & \text{for } -2\pi < x < 0 \\ -\frac{1}{2} + \frac{1}{2}\sin\left(\frac{x}{2}\right) & \text{for } 0 < x < 4\pi \end{cases}$$

4 : $\begin{cases} 1 : \frac{d}{dx}\left(\cos\left(\frac{x}{2}\right)\right) \\ 1 : g'(x) \\ 1 : x = 0 \\ 1 : x = \pi \end{cases}$

$f'(x)$ does not exist at $x = 0$.

For $-2\pi < x < 0$, $f'(x) \neq 0$.

For $0 < x < 4\pi$, $f'(x) = 0$ when $x = \pi$.

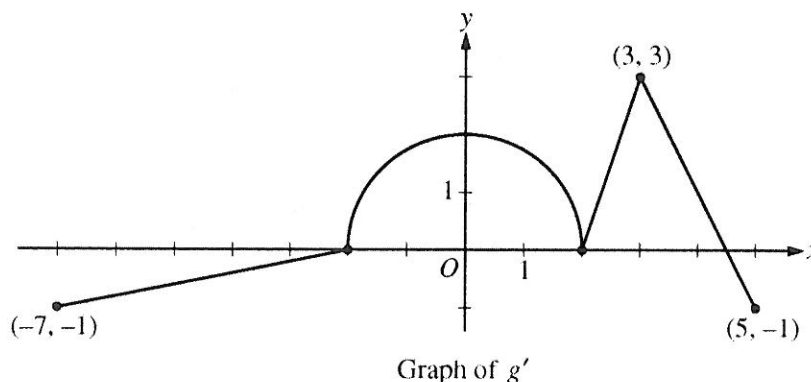
f has critical points at $x = 0$ and $x = \pi$.

$$\begin{aligned} \text{(c)} \quad h'(x) &= g(3x) \cdot 3 \\ h'\left(-\frac{\pi}{3}\right) &= 3g(-\pi) = 3\pi \end{aligned}$$

3 : $\begin{cases} 2 : h'(x) \\ 1 : \text{answer} \end{cases}$

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Question 5



The function g is defined and differentiable on the closed interval $[-7, 5]$ and satisfies $g(0) = 5$. The graph of $y = g'(x)$, the derivative of g , consists of a semicircle and three line segments, as shown in the figure above.

- (a) Find $g(3)$ and $g(-2)$.
- (b) Find the x -coordinate of each point of inflection of the graph of $y = g(x)$ on the interval $-7 < x < 5$. Explain your reasoning.
- (c) The function h is defined by $h(x) = g(x) - \frac{1}{2}x^2$. Find the x -coordinate of each critical point of h , where $-7 < x < 5$, and classify each critical point as the location of a relative minimum, relative maximum, or neither a minimum nor a maximum. Explain your reasoning.

(a) $g(3) = 5 + \int_0^3 g'(x) dx = 5 + \frac{\pi \cdot 2^2}{4} + \frac{3}{2} = \frac{13}{2} + \pi$
 $g(-2) = 5 + \int_0^{-2} g'(x) dx = 5 - \pi$

3 : $\begin{cases} 1 : \text{uses } g(0) = 5 \\ 1 : g(3) \\ 1 : g(-2) \end{cases}$

- (b) The graph of $y = g(x)$ has points of inflection at $x = 0$, $x = 2$, and $x = 3$ because g' changes from increasing to decreasing at $x = 0$ and $x = 3$, and g' changes from decreasing to increasing at $x = 2$.

2 : $\begin{cases} 1 : \text{identifies } x = 0, 2, 3 \\ 1 : \text{explanation} \end{cases}$

(c) $h'(x) = g'(x) - x = 0 \Rightarrow g'(x) = x$

On the interval $-2 \leq x \leq 2$, $g'(x) = \sqrt{4 - x^2}$.

On this interval, $g'(x) = x$ when $x = \sqrt{2}$.

The only other solution to $g'(x) = x$ is $x = 3$.

$h'(x) = g'(x) - x > 0$ for $0 \leq x < \sqrt{2}$

$h'(x) = g'(x) - x \leq 0$ for $\sqrt{2} < x \leq 5$

Therefore h has a relative maximum at $x = \sqrt{2}$, and h has neither a minimum nor a maximum at $x = 3$.

4 : $\begin{cases} 1 : h'(x) \\ 1 : \text{identifies } x = \sqrt{2}, 3 \\ 1 : \text{answer for } \sqrt{2} \text{ with analysis} \\ 1 : \text{answer for 3 with analysis} \end{cases}$

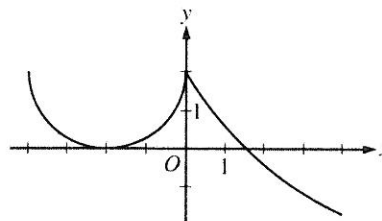
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2009 SCORING GUIDELINES

Question 6

The derivative of a function f is defined by

$$f'(x) = \begin{cases} g(x) & \text{for } -4 \leq x \leq 0 \\ 5e^{-x/3} - 3 & \text{for } 0 < x \leq 4 \end{cases}$$

The graph of the continuous function f' , shown in the figure above, has x -intercepts at $x = -2$ and $x = 3\ln\left(\frac{5}{3}\right)$. The graph of g on $-4 \leq x \leq 0$



Graph of f'

is a semicircle, and $f(0) = 5$.

- (a) For $-4 < x < 4$, find all values of x at which the graph of f has a point of inflection. Justify your answer.
- (b) Find $f(-4)$ and $f(4)$.
- (c) For $-4 \leq x \leq 4$, find the value of x at which f has an absolute maximum. Justify your answer.

- (a) f' changes from decreasing to increasing at $x = -2$ and from increasing to decreasing at $x = 0$. Therefore, the graph of f has points of inflection at $x = -2$ and $x = 0$.

2: $\begin{cases} 1: \text{identifies } x = -2 \text{ or } x = 0 \\ 1: \text{answer with justification} \end{cases}$

(b)
$$f(-4) = 5 + \int_0^{-4} g(x) dx$$

$$= 5 - (8 - 2\pi) = 2\pi - 3$$

$$f(4) = 5 + \int_0^4 (5e^{-x/3} - 3) dx$$

$$= 5 + \left(-15e^{-x/3} - 3x\right)\bigg|_{x=0}^{x=4}$$

$$= 8 - 15e^{-4/3}$$

5: $\begin{cases} 2: f(-4) \\ 1: \text{integral} \\ 1: \text{value} \\ 3: f(4) \\ 1: \text{integral} \\ 1: \text{antiderivative} \\ 1: \text{value} \end{cases}$

- (c) Since $f'(x) > 0$ on the intervals $-4 < x < -2$ and $-2 < x < 3\ln\left(\frac{5}{3}\right)$, f is increasing on the interval $-4 \leq x \leq 3\ln\left(\frac{5}{3}\right)$.

2: $\begin{cases} 1: \text{answer} \\ 1: \text{justification} \end{cases}$

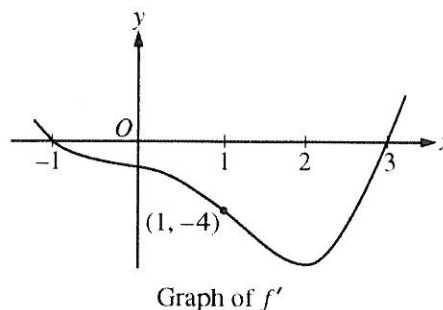
Since $f'(x) < 0$ on the interval $3\ln\left(\frac{5}{3}\right) < x < 4$, f is decreasing on the interval $3\ln\left(\frac{5}{3}\right) \leq x \leq 4$.

Therefore, f has an absolute maximum at $x = 3\ln\left(\frac{5}{3}\right)$.

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2009 SCORING GUIDELINES (Form B)

Question 5

Let f be a twice-differentiable function defined on the interval $-1.2 < x < 3.2$ with $f(1) = 2$. The graph of f' , the derivative of f , is shown above. The graph of f' crosses the x -axis at $x = -1$ and $x = 3$ and has a horizontal tangent at $x = 2$. Let g be the function given by $g(x) = e^{f(x)}$.



- (a) Write an equation for the line tangent to the graph of g at $x = 1$.
- (b) For $-1.2 < x < 3.2$, find all values of x at which g has a local maximum. Justify your answer.
- (c) The second derivative of g is $g''(x) = e^{f(x)}[(f'(x))^2 + f''(x)]$. Is $g''(-1)$ positive, negative, or zero? Justify your answer.
- (d) Find the average rate of change of g' , the derivative of g , over the interval $[1, 3]$.

- (a) $g(1) = e^{f(1)} = e^2$
 $g'(x) = e^{f(x)}f'(x)$, $g'(1) = e^{f(1)}f'(1) = -4e^2$
 The tangent line is given by $y = e^2 - 4e^2(x - 1)$.

- 3 : $\begin{cases} 1 : g'(x) \\ 1 : g(1) \text{ and } g'(1) \\ 1 : \text{tangent line equation} \end{cases}$

- (b) $g'(x) = e^{f(x)}f'(x)$
 $e^{f(x)} > 0$ for all x
 So, g' changes from positive to negative only when f' changes from positive to negative. This occurs at $x = -1$ only. Thus, g has a local maximum at $x = -1$.

- 2 : $\begin{cases} 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

- (c) $g''(-1) = e^{f(-1)}[(f'(-1))^2 + f''(-1)]$
 $e^{f(-1)} > 0$ and $f'(-1) = 0$
 Since f' is decreasing on a neighborhood of -1 , $f''(-1) < 0$. Therefore, $g''(-1) < 0$.

- 2 : $\begin{cases} 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

- (d) $\frac{g'(3) - g'(1)}{3 - 1} = \frac{e^{f(3)}f'(3) - e^{f(1)}f'(1)}{2} = 2e^2$

- 2 : $\begin{cases} 1 : \text{difference quotient} \\ 1 : \text{answer} \end{cases}$